Chapter 5. Fisher's Exact P-Values for Completely Randomized Experiments

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July 15, 2022

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Contents

5.1 Introduction

Fisher's Exact P-Values : FEPs

1. **The sharp(or exact) null hypothesis**(Fisher, 1935)
   - **(T)** : Test Statistic
   - **(Y_i^{obs})** : The observed outcomes→ Y(0), Y(1)
   - **(W)** : A function of the stochastic assignment vector
   - **(X)** :Any pre-treatment variables
2. **Fisher Exact P-values(FEPs)** : Two steps
   - The choice of a sharp null hypothesis
   - The choice of test statistic
3. **Data Set** : Honey Experiment Data for Coughing Children
   - Calculating FEPs(Choice of null hypothesis and Test statistic)

5.2 The Paul Honey Experiment data

Data : a randomized experiment of three treatments

- Three treatments are :
    i . **single dose of buckwheat honey ★**
    iii . A single dose of honey-flavored dex-tromethorphan
    iii . **no active treatment ★**

- N=72 → N_t=35(buckwheat honey), N_t=37(no treatment)
    - Variable(cfa, csa, csp, csa) : Cough Frequency and Severity
    - Outcome Scale : 0 to 6

5.3. A Simple Example with Six Units(1/3)

<A subsample from the honey data set, with 6 children>

Table 5.3. ***Cough Frequency for the First Six Units from the Honey Study***

Unit	Potential Outcomes				
	Cough Frequency (cfa)		Observed Variables		
	$Y_i(0)$	$Y_i(1)$	W_i	X_i (cfp)	Y_i^{obs} (cfa)
1	?	3	1	4	3
2	?	5	1	6	5
3	?	0	1	4	0
4	4	?	0	4	4
5	0	?	0	1	0
6	1	?	0	5	1

- Fundamental problem of casual inference shown on Table 5.3
    - $W_i = 1$: treatment group, ${Y_i}^{obs}$, ${X_i}^{obs}$: cfp
    - Problems : Many of the potential outcomes are **missing**

5.3. A Simple Example with Six Units(2/3)

<A subsample from the honey data set, with 6 children>

- $H_0 : Y_i(0) = Y_i(1) \rightarrow$ **Null hypothesis**
  - The treatment had no effect on coughing outcomes
  - the missing outcomes $Y_i^{mis} = Y_i^{obs}$
  - By using the observed data, we can fill in all 6 **'?'**
- By Null hypothesis we can fill in :

Unit	Potential Outcomes					
	Cough Frequency (cfa)		Observed Variables			
	$Y_i(0)$	$Y_i(1)$	Treatment	X_i	Y_i^{obs}	rank(Y_i^{obs})
1	(3)	3	1	4	3	4
2	(5)	5	1	6	5	6
3	(0)	0	1	4	0	1.5
4	4	(4)	0	4	4	5
5	0	(0)	0	1	0	1.5
6	1	(1)	0	5	1	3

5.3. A Simple Example with Six Units(3/3)

<A subsample from the honey data set, with 6 children>

- $T\left(\mathbf{W}, \mathbf{Y}^{\text{obs}}\right) = T^{\text{dif}} = \left|\bar{Y}_{\text{t}}^{\text{obs}} - \bar{Y}_{\text{c}}^{\text{obs}}\right|$
  - $\bar{Y}_{\text{t}}^{\text{obs}} = \sum_{i:W_i=1} Y_i^{\text{obs}}/N_{\text{t}}$ and $\bar{Y}_{\text{c}}^{\text{obs}} = \sum_{i:W_i=0} Y_i^{\text{obs}}/N_{\text{c}}$
  - $N_{\text{c}} = \sum_{i=1}^{N}(1 - W_i)$ and $N_{\text{t}} = \sum_{i=1}^{N} W_i$
- Each vector of treatment assignments, W does not change the values of outcomes
  - $T\left(\mathbf{W}, \mathbf{Y}^{\text{obs}}\right)$ varies with W, 20 possible vectors

Table 5.5. ***Randomization Distribution for Two Statistics for the Honey Data from Table 5.3***

						Statistic: Absolute Value of Difference in Average	
W_1	W_2	W_3	W_4	W_5	W_6	Levels (Y_i)	Ranks (R_i)
					⋮		
1	1	0	0	0	1	1.67	1.67
1	1	0	0	1	0	1.00	0.67
1	1	0	1	0	0	3.67	3.00
1	1	1	0	0	0	**1.00**	**0.67**

5.4 The Choice of Null hypothesis

Fisher only focused on what is the most obvious null hypothesis, **that of no effect whatsoever of the active treatment**

- $H_0 : Y_i(0) = Y_i(1) \rightarrow$ **Null hypothesis**
  - The first choice when calculating the FEP is the choice of null hypothesis
  - The null hypothesis is that of no effect whatsoever $Y_i(0) = Y_i(1)$, $Y_i^{mis} = Y_i^{obs}$

5.5 The Choice of Statistic

The choice of test statistic is more difficult than the choice of the null hypothesis

1. **Test statistic** : to find a p-value under the null hypothesis
    - **Transformations**
        - Attractive option when it comes to constant multiplicative effect of the treatment
        - $T^{log} = \left|\sum_{i:W_i=1} ln(Y_i^{obs})/N_t - \sum_{i:W_i=0} ln(Y_i^{obs})/N_c\right|$
    - **T-Statistics**
        - Equal means, with unequal variances in the two groups
        - $T^{t-stat} = \left|\overline{Y_t^{obs}} - \overline{Y_c^{obs}}/\sqrt{s_c^2/N_c + s_t^2/N_t}\right|$
    - **Rank-Statistics**
        - Transforming the outcomes to ranks
        - $T^{rank} = \left|\overline{R^t} - \overline{R^c}\right| = \left|\sum_{i:W_i=1} R_i/N_t - \sum_{i:W_i=0} R_i/N_c\right|$
        - Use when the distribution of raw outcomes has a substantial number of outliers
        - Ranks are related to the indexed list of order statistics

- **Three different test statistics**
  - $T^{dif}, T^{med}, T^{rank}$: To see how much power they had
  - Rank-based statistics is an attractive model and the others play to their advantages according to the settings
  - $Y_i(0) = Y_i(1) + \tau$, τ : treatment effect

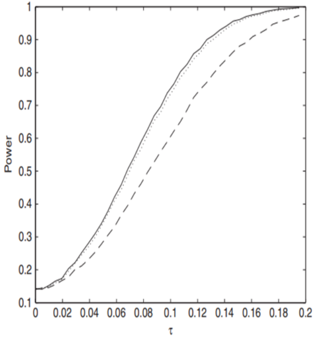

Figure 5.1. Additive model with normal outcomes T^{dif}(-), T^{median}(—), T^{rank}(...)

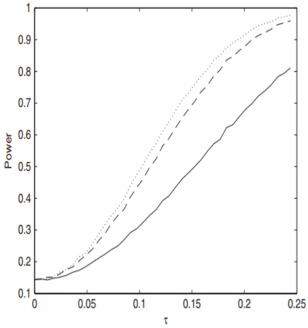

Figure 5.2. Additive model with outliers T^{dif}(-), T^{median}(—), T^{rank}(...)

- **Using the pre-treatment variables, Covariates :** X_i
  - $T\left(\mathbf{W}, \mathbf{Y}^{\text{obs}}\right) = T^{\text{dif}} = \left|\bar{Y}_{\text{t}}^{\text{obs}} - \bar{Y}_{\text{c}}^{\text{obs}}\right|$
  - $T(\mathbf{W}, \mathbf{Y}^{\text{obs}}, \mathbf{X}) = \overline{Y_t}^{obs} - \overline{Y_c}^{obs} - (\overline{X_t} - \overline{X_c})$
- **P-Values for Honey Data Using Various Statistics**

Test Statistic	Statistic	P-Value
T^{dif}	−0.697	0.067
T^{quant} $(\delta = 0.25)$	−1.000	0.440
T^{quant} $(\delta = 0.50)$	−1.000	0.637
T^{quant} $(\delta = 0.75)$	−1.000	0.576
$T^{\text{t-stat}}$	−1.869	0.065
T^{rank}	−9.785	0.043
T^{ks}	0.304	0.021
$T^{\text{F-stat}}$	3.499	0.182
T^{gain}	−0.967	0.006
$T^{\text{reg-coef}}$	−0.911	0.008

5.11 Conclusion

- **FEP approach**
  : To assess the premise of a sharp null hypothesis
    - Compared to Chi Squared method, FEP is used when samples are small such as 30 samples
    - Under the null hypothesis of absolutely no effect of the treatment, calculate the p-value.